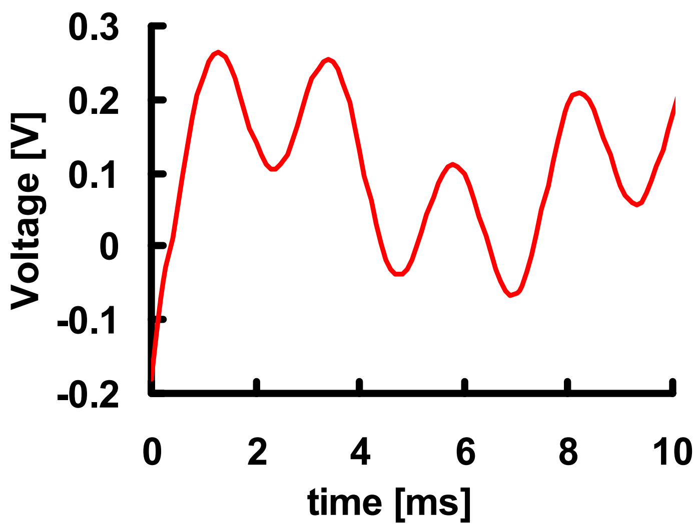
Analog & digital signals

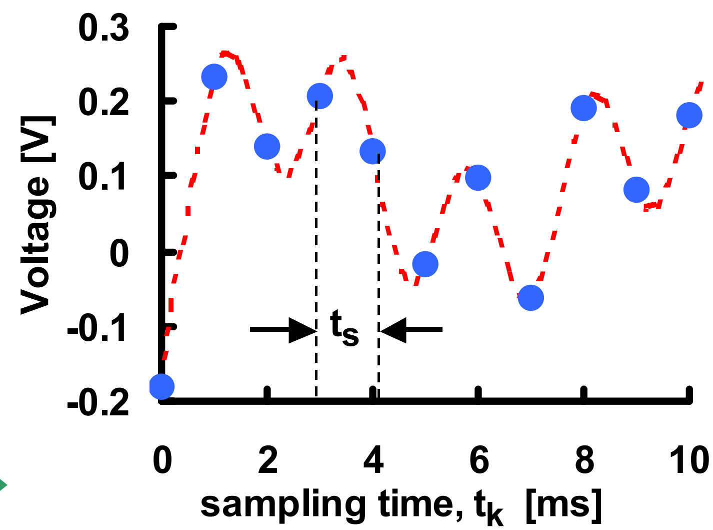
Analog

Continuous function V of *continuous* variable t (time, space etc) : V(t).

Digital

Discrete function V_k of *discrete* sampling variable t_k, with k = integer: $V_k = V(t_k)$.





Uniform (periodic) sampling.
Sampling frequency $f_S = 1/ t_S$

Slides adapted from ME Angoletta, CERN

Digital vs analog proc'ing

Digital Signal Processing (DSPing)

Advantages

- More flexible.
- Often easier system upgrade.
- Data easily stored.
- Better control over accuracy requirements.
- Reproducibility.

Limitations

- A/D & signal processors speed: wide-band signals still difficult to treat (real-time systems).
- Finite word-length effect.
- Obsolescence (analog electronics has it, too!).

Digital system example

General scheme !

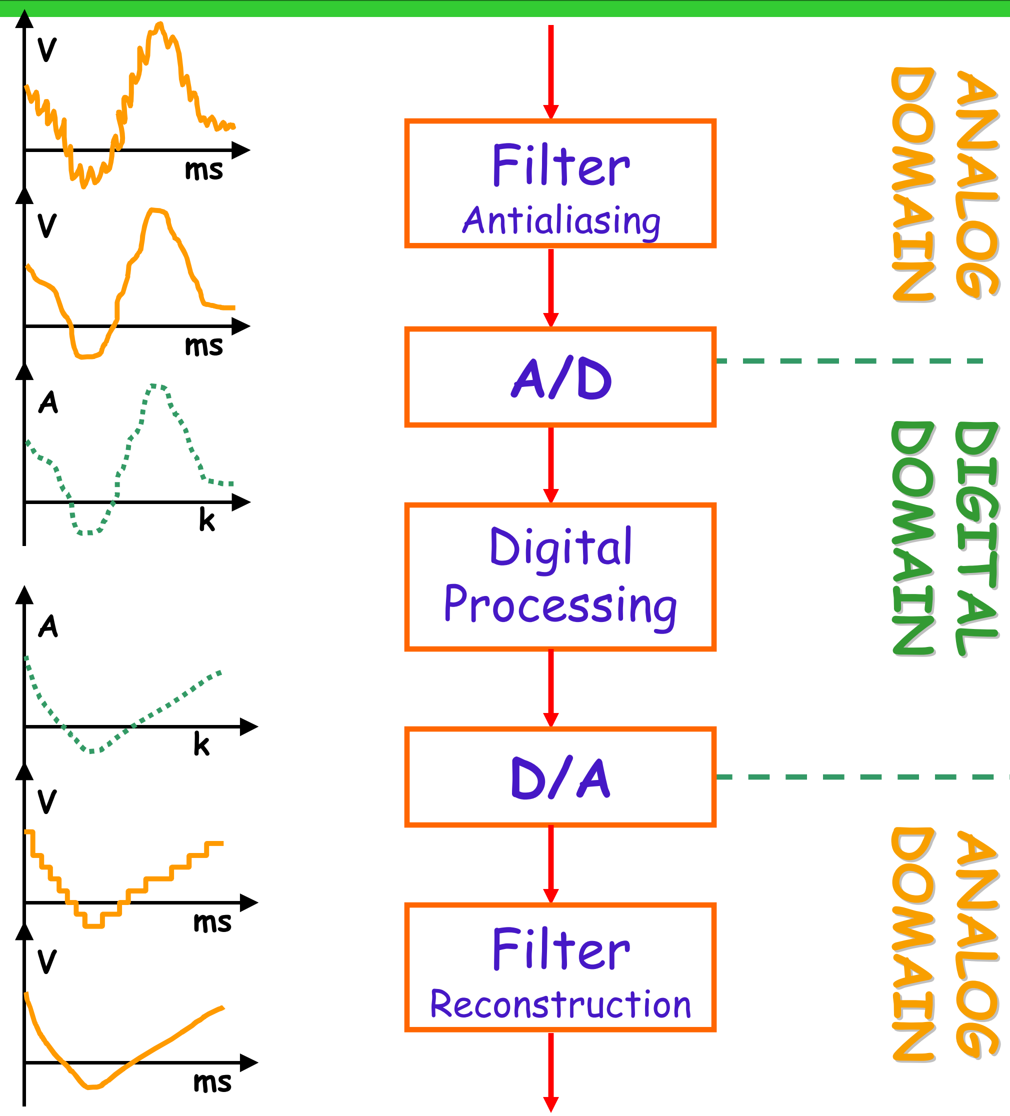
Sometimes steps missing

- Filter + A/D

(ex: economics);

- D/A + filter

(ex: digital output wanted).



Digital system implementation

ANALOG INPUT

KEY DECISION POINTS:
Analysis bandwidth, Dynamic range

↓

Antialiasing Filter

↓

A/D

↓

Digital Processing

↓

DIGITAL OUTPUT

- Pass / stop bands.
- Sampling rate.

  (1)

- No. of bits. Parameters.

  (2)

- Digital format.

  (3)

What to use for processing?
See slide "DSPing aim & tools"

① Sampling

How fast must we sample * a continuous signal to preserve its info content?

Ex: train wheels in a movie.

25 frames (=samples) per second.

Train starts ➡ wheels 'go' clockwise.

Train accelerates ➡ wheels 'go' counter-clockwise.

Why?

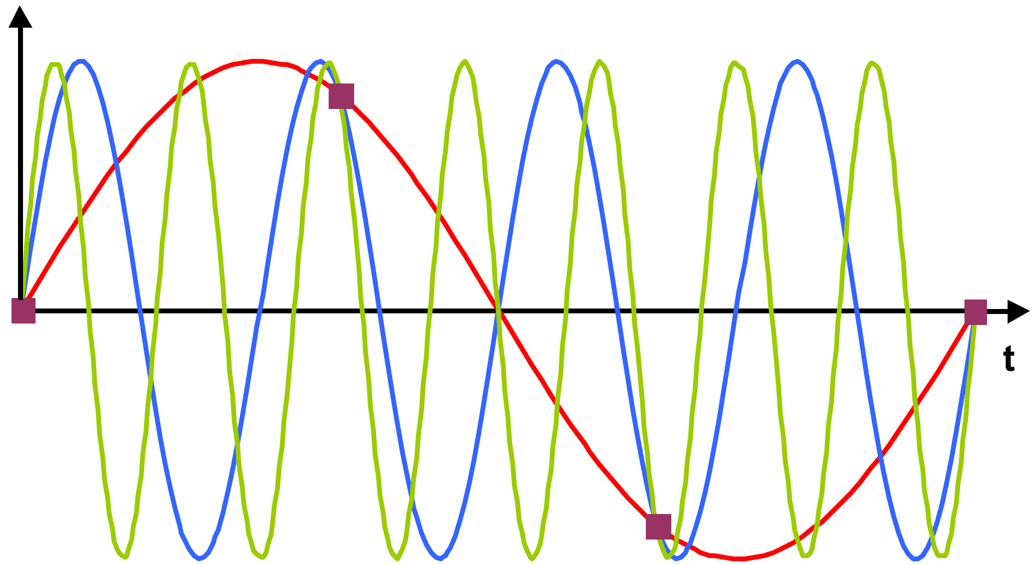
Frequency misidentification due to low sampling frequency.

* Sampling: independent variable (ex: time) continuous → discrete.

Quantisation: dependent variable (ex: voltage) continuous → discrete.

Here we'll talk about uniform sampling.

① Sampling - 2



— $s(t) = \sin(2\pi f_0 t)$

■ $s(t)$ @ f_S

$f_0 = 1$ Hz, $f_S = 3$ Hz

— $s_1(t) = \sin(8\pi f_0 t)$

— $s_2(t) = \sin(14\pi f_0 t)$

$s(t)$ @ f_S represents exactly all sine-waves $s_k(t)$ defined by:

$$s_k(t) = \sin(2\pi (f_0 + k f_S) t), \quad |k| \in \mathbb{N}$$

① The sampling theorem

Theo* A signal s(t) with maximum frequency f_{MAX} can be recovered if sampled at frequency $f_S > 2\ f_{MAX}$.

* Multiple proposers: Whittaker(s), Nyquist, Shannon, Kotel'nikov.

Naming gets confusing !

Nyquist frequency (rate) $f_N = 2\ f_{MAX}$ *or* f_{MAX} *or* $f_{S,MIN}$ *or* $f_{S,MIN}/2$

Example

$$s(t) = \underbrace{3 \cdot \cos(50\pi t)}_{F_1} + \underbrace{10 \cdot \sin(300\pi t)}_{F_2} - \underbrace{\cos(100\pi t)}_{F_3}$$

Condition on f_S?

F_1=25 Hz, F_2 = 150 Hz, F_3 = 50 Hz

f_{MAX} = F_2

$f_S > 300$ Hz

① Frequency domain (hints)

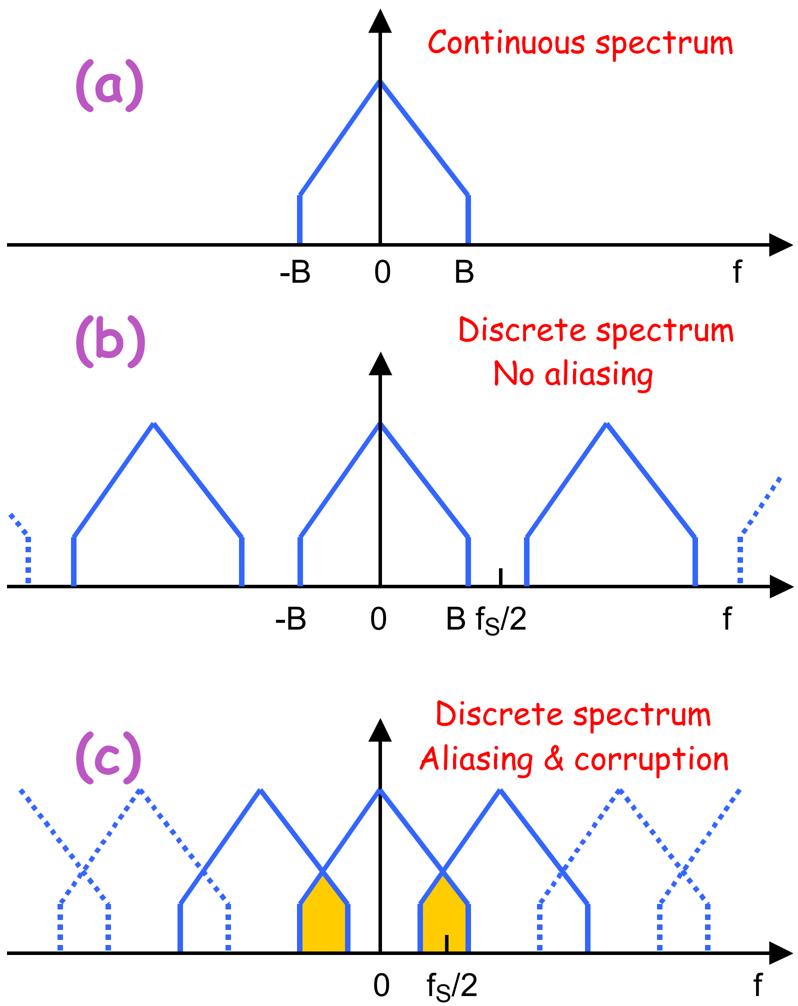
- **Time & frequency**: two complementary signal descriptions. Signals seen as “projected’ onto time or frequency domains.

Example

Ear + brain act as frequency analyser: audio spectrum split into many narrow bands ⇨ low-power sounds detected out of loud background.

- **Bandwidth**: indicates rate of change of a signal. High bandwidth ⇨ signal changes fast.

① Sampling low-pass signals



(a) Band-limited signal: frequencies in [-B, B] (f_{MAX} = B).

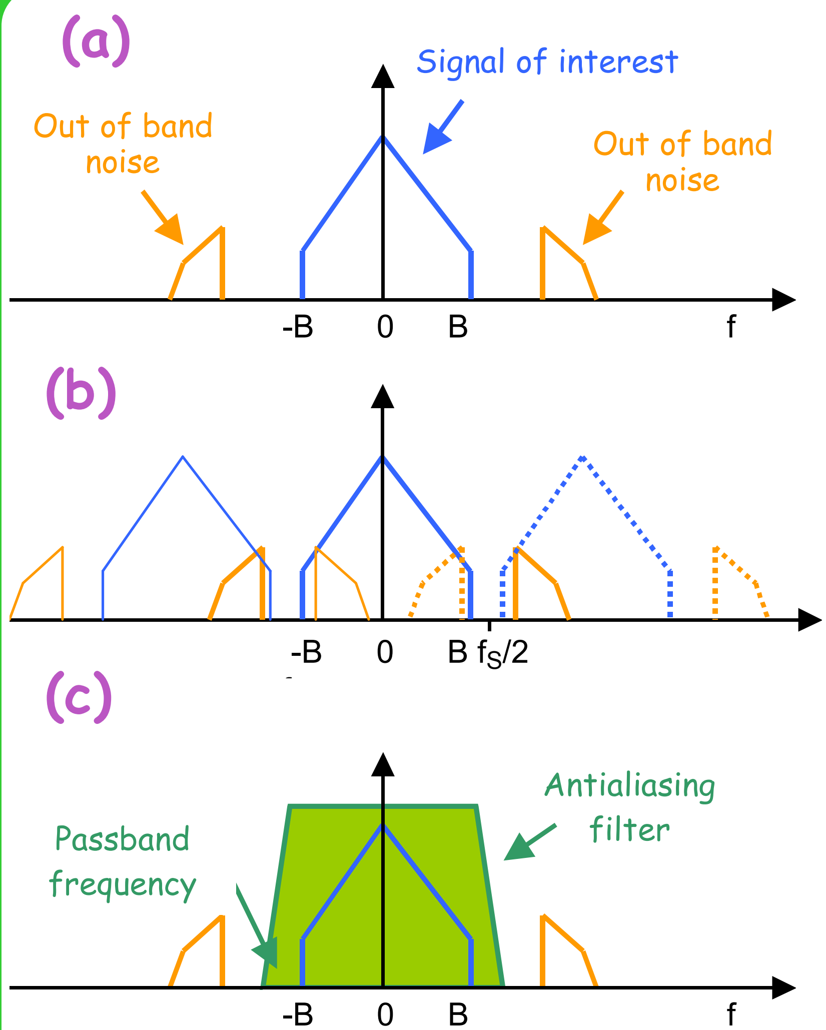
(b) Time sampling ➡ frequency repetition.

$f_S > 2B$ ➡ no aliasing.

(c) $f_S \leqslant 2B$ ➡ *aliasing !*

Aliasing: signal ambiguity in frequency domain

① Antialiasing filter



(a),(b) *Out-of-band* noise can alias into band of interest. **Filter it before!**

(c) Antialiasing filter

Passband: depends on bandwidth of interest.

Attenuation A_{MIN} : depends on

- ADC resolution (number of bits N).

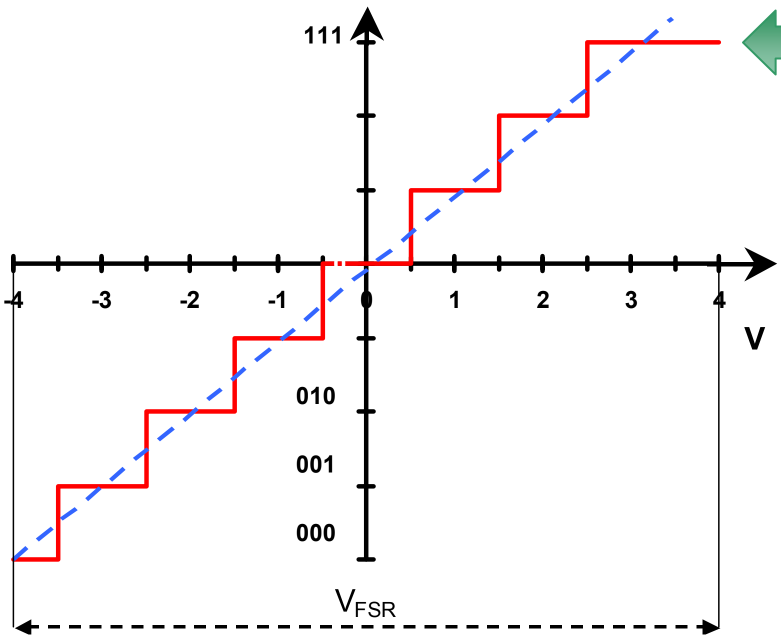
$$A_{MIN,\ dB} \sim 6.02\ N + 1.76$$

- Out-of-band noise magnitude.

Other parameters: ripple, stopband frequency...

② ADC - Number of bits N

Continuous input signal digitized into **2^N levels.**

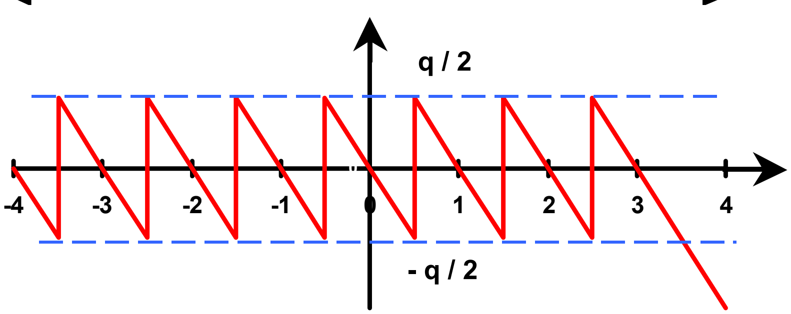


Uniform, bipolar transfer function (N=3)

<u>Quantization step</u> $q = \frac{V_{FSR}}{2^N}$

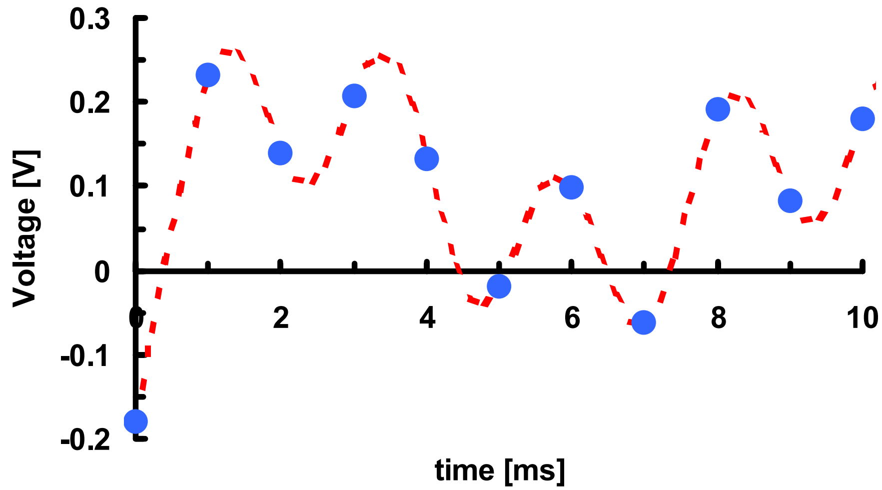
Ex: V_{FSR} = 1V , N = 12 ➡ q = 244.1 μV

<u>LSB</u>
- Voltage (= q)
- Scale factor (= $1 / 2^N$)
- Percentage (= $100 / 2^N$)



Quantisation error

② ADC - Quantisation error

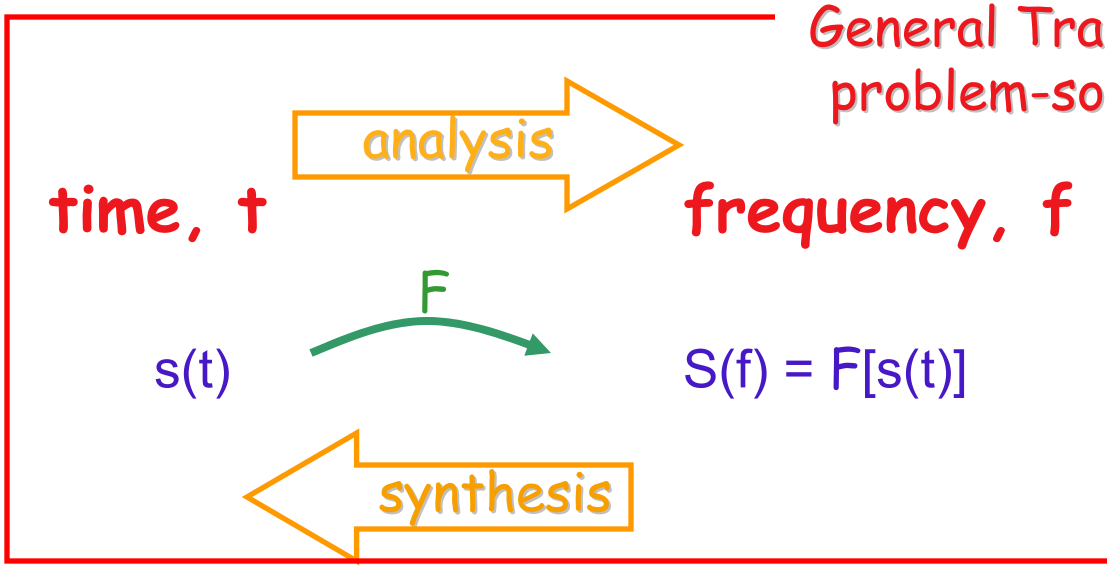


- Quantisation Error e_q in [-0.5 q, +0.5 q].
- e_q limits ability to resolve small signal.
- Higher resolution means lower e_q.

Frequency analysis: why?

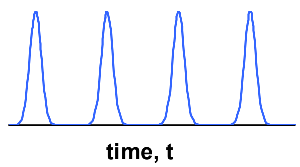
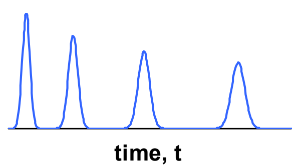
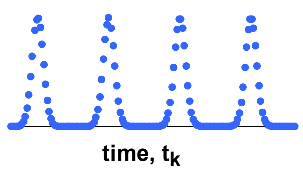
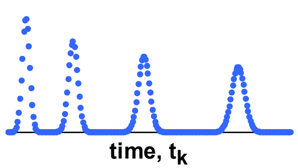
- Fast & efficient insight on signal's building blocks.
- Simplifies original problem - ex.: solving Part. Diff. Eqns. (PDE).
- Powerful & complementary to time domain analysis techniques.
- The brain does it?

General Transform as problem-solving tool



s(t), S(f) :
Transform Pair

Fourier analysis - tools

Input Time Signal				Frequency spectrum	
	Continuous	Periodic *(period T)*	FS	Discrete	$c_k = \frac{1}{T} \cdot \int_0^T s(t) \cdot e^{-jk\omega t} dt$
		Aperiodic	FT	Continuous	$S(f) = \int_{-\infty}^{+\infty} s(t) \cdot e^{-j2\pi f t} dt$
	Discrete	Periodic *(period T)*	DFS**	Discrete	$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} s[n] \cdot e^{-j\frac{2\pi k n}{N}}$
		Aperiodic	DTFT	Continuous	$S(f) = \sum_{n=-\infty}^{+\infty} s[n] \cdot e^{-j2\pi f n}$
			DFT**	Discrete	$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} s[n] \cdot e^{-j\frac{2\pi k n}{N}}$

Note: $j = \sqrt{-1}$, $\omega = 2\pi/T$, $s[n]=s(t_n)$, N = No. of samples

** *Calculated via FFT*

Slides adapted from ME Angoletta, CERN

A little history

- Astronomic predictions by Babylonians/Egyptians likely via trigonometric sums.
- **1669**: Newton stumbles upon light spectra (*specter* = ghost) but fails to recognise "frequency" concept (*corpuscular* theory of light, & no waves).
- **18th century**: two outstanding problems
  - → celestial bodies orbits: Lagrange, Euler & Clairaut approximate observation data with linear combination of periodic functions; Clairaut,1754(!) first DFT formula.
  - → vibrating strings: Euler describes vibrating string motion by sinusoids (wave equation).
- **1807**: Fourier presents his work on heat conduction ⇒ Fourier analysis born.
  - → Diffusion equation ⇔ series (infinite) of sines & cosines. Strong criticism by peers blocks publication. Work published, 1822 (*"Theorie Analytique de la chaleur"*).

A little history -2

- **19th / 20th century**: two paths for Fourier analysis - Continuous & Discrete.

CONTINUOUS

→ Fourier extends the analysis to arbitrary function (Fourier Transform).

→ Dirichlet, Poisson, Riemann, Lebesgue address FS convergence.

→ Other FT variants born from varied needs (ex.: Short Time FT - speech analysis).

DISCRETE: Fast calculation methods (FFT)

→ **1805** - Gauss, first usage of FFT (manuscript in Latin went unnoticed!!! Published 1866).

→ **1965** - IBM's Cooley & Tukey "rediscover" FFT algorithm (*"An algorithm for the machine calculation of complex Fourier series"*).

→ Other DFT variants for different applications (ex.: Warped DFT - filter design & signal compression).

→ FFT algorithm refined & modified for most computer platforms.

Fourier Series (FS)

A periodic function s(t) satisfying Dirichlet's conditions * can be expressed as a Fourier series, with harmonically related sine/cosine terms.

synthesis

$$s(t) = a_0 + \sum_{k=1}^{+\infty} \left[a_k \cdot \cos(k \omega t) - b_k \cdot \sin(k \omega t) \right]$$

For all t but discontinuities

a_0, a_k, b_k : Fourier coefficients.

k: harmonic number,

T: period, $\omega = 2\pi/T$

analysis

$$a_0 = \frac{1}{T} \cdot \int_0^T s(t) dt$$

(signal average over a period, i.e. DC term & zero-frequency component.)

$$a_k = \frac{2}{T} \cdot \int_0^T s(t) \cdot \cos(k \omega t) dt$$

$$-b_k = \frac{2}{T} \cdot \int_0^T s(t) \cdot \sin(k \omega t) dt$$

Note: $\{\cos(k\omega t), \sin(k\omega t)\}_k$ form orthogonal base of function space.

* see next slide

FS convergence

Dirichlet conditions

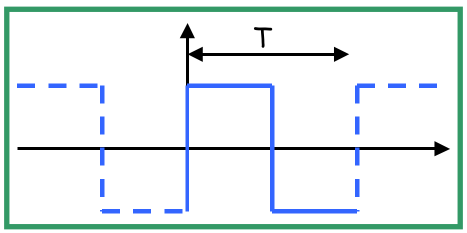
In any period:

(a) s(t) piecewise-continuous;

(b) s(t) piecewise-monotonic;

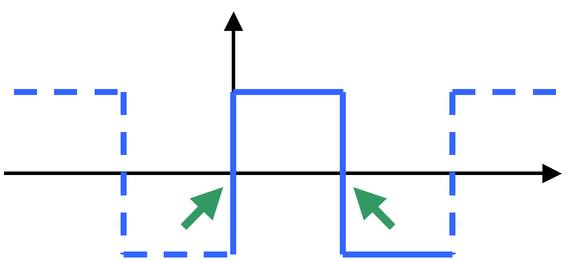
(c) s(t) absolutely integrable , $\int_0^T |s(t)|\, dt < \infty$

Example: square wave

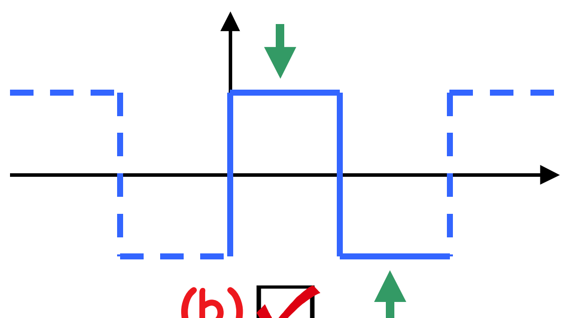


Rate of convergence

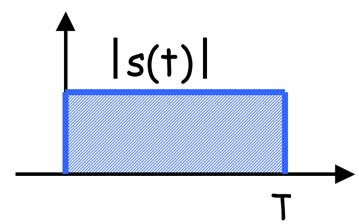
if s(t) discontinuous then $|a_k| < M/k$ for large k (M>0)



(a) ☑



(b) ☑



(c) ☑

FS analysis - 1

FS of odd* function: square wave.

$T = 2\pi \Rightarrow \omega = 1$

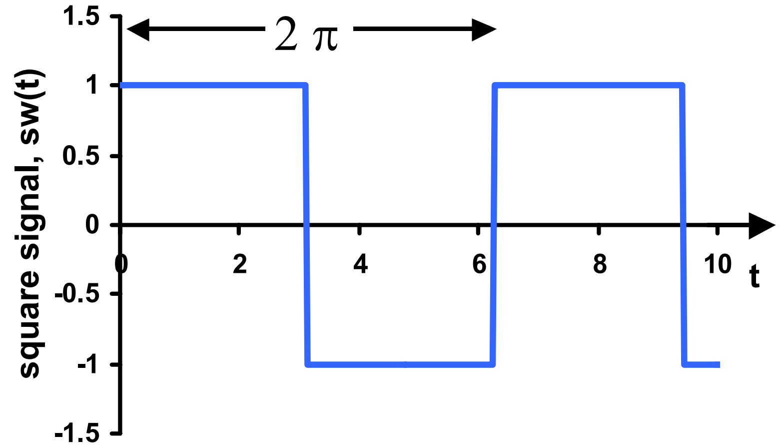
$$a_0 = \frac{1}{2\pi}\cdot\left\{\int_0^{\pi} dt + \int_{\pi}^{2\pi}(-1)dt\right\} = 0$$ *(zero average)*

$$a_k = \frac{1}{\pi}\cdot\left\{\int_0^{\pi} \cos kt\, dt - \int_{\pi}^{2\pi} \cos kt\, dt\right\} = 0$$ *(odd function)*

$$-b_k = \frac{1}{\pi}\cdot\left\{\int_0^{\pi} \sin kt\, dt - \int_{\pi}^{2\pi} \sin kt\, dt\right\} = \ldots = \frac{2}{k\cdot\pi}\cdot\{1-\cos k\pi\} =$$

$$= \begin{cases} \dfrac{4}{k\cdot\pi} & , k \text{ odd} \\ 0 & , k \text{ even} \end{cases}$$

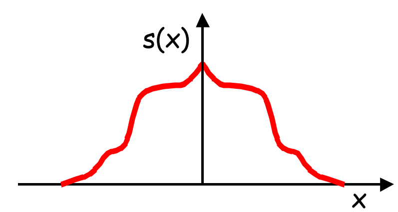
$$sw(t) = \frac{4}{\pi}\cdot\sin t + \frac{4}{3\cdot\pi}\cdot\sin 3\cdot t + \frac{4}{5\cdot\pi}\cdot\sin 5\cdot t + \ldots$$



* Even & Odd functions

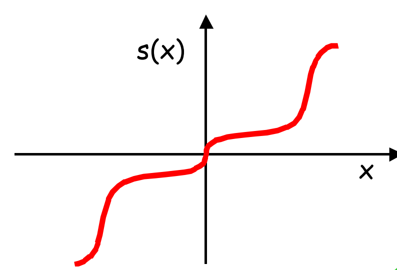
Even :

$s(-x) = s(x)$



Odd :

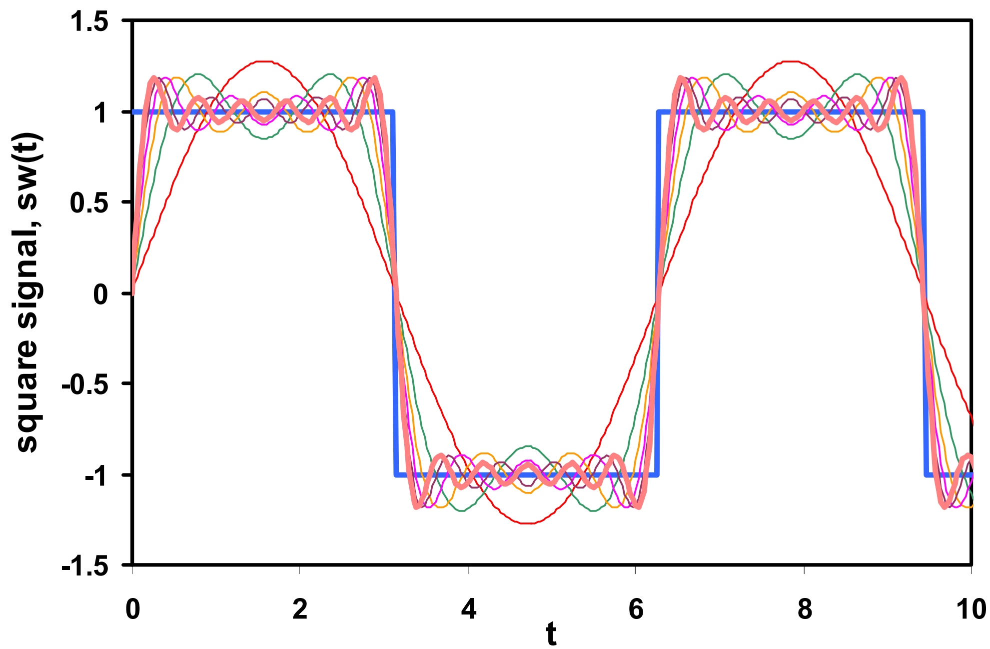
$s(-x) = -s(x)$



FS synthesis

Square wave reconstruction from spectral terms

$$sw(t) = \sum_{k=1}^{1} \left[b_k \cdot \sin(kt) \right]$$



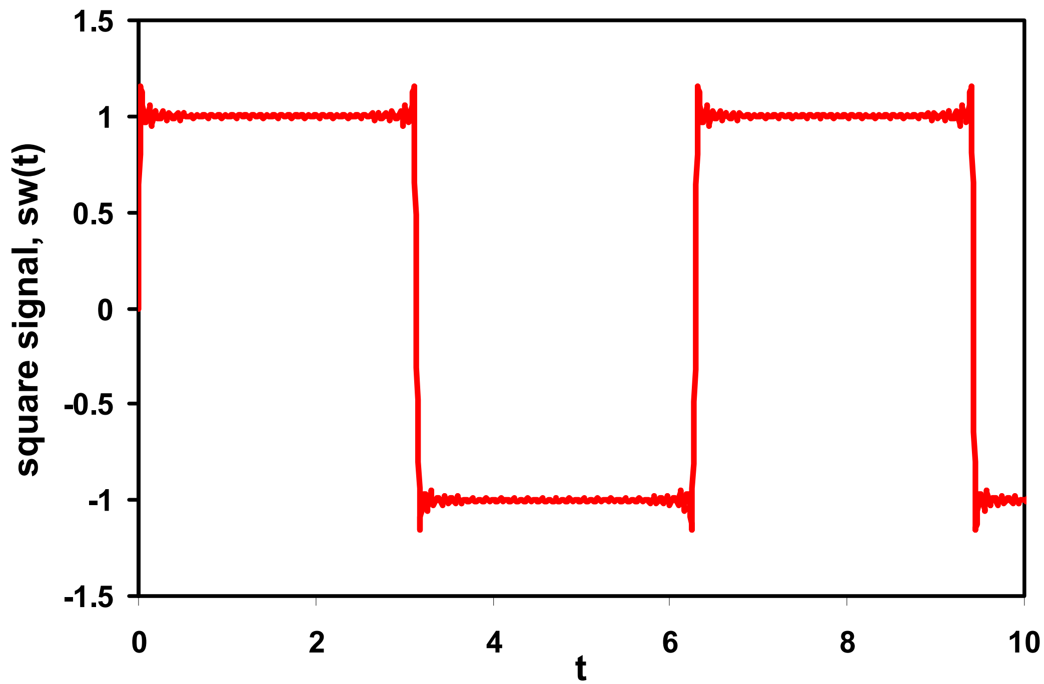
Convergence may be slow (~1/k) - ideally need infinite terms.

Practically, series truncated when remainder below computer tolerance (⇒ *error*). ***BUT*** … Gibbs' Phenomenon.

Gibbs phenomenon

Overshoot exist @ each discontinuity

$$sw_{79}(t) = \sum_{k=1}^{79} \left[-b_k \cdot \sin(kt) \right]$$



- First observed by Michelson, 1898. Explained by Gibbs.
- Max overshoot pk-to-pk = 8.95% of discontinuity magnitude. Just a minor annoyance.
- FS converges to (-1+1)/2 = 0 @ discontinuities, *in this case*.

FS time shifting

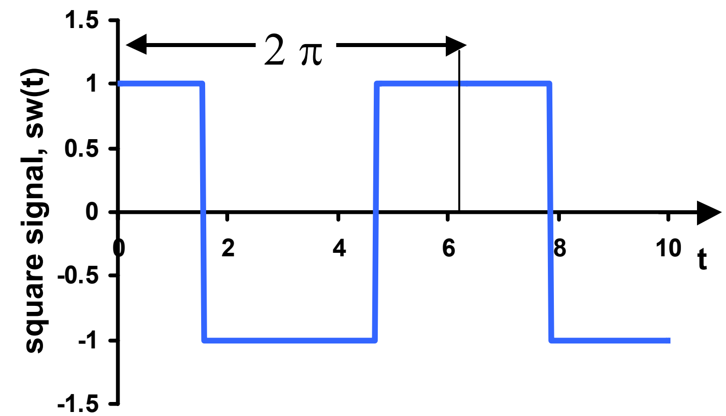
FS of even function:
$\pi/2$-advanced square-wave

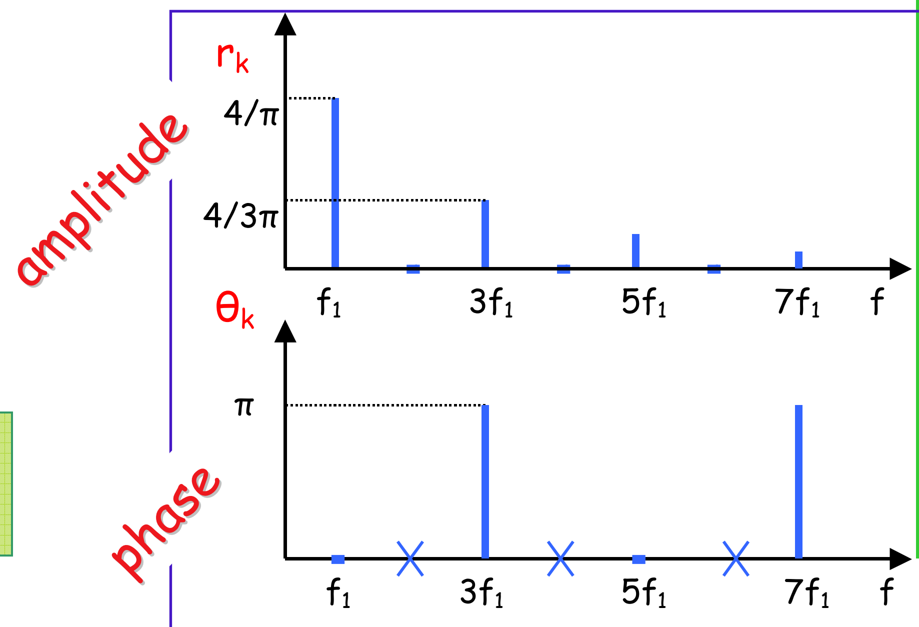
$a_0 = 0$ *(zero average)*

$$a_k = \begin{cases} \frac{4}{k\cdot\pi}, & k \text{ odd}, \quad k = 1, 5, 9... \\ -\frac{4}{k\cdot\pi}, & k \text{ odd}, \quad k = 3, 7, 11... \\ 0, & k \text{ even.} \end{cases}$$

$-b_k = 0$ *(even function)*

Note: amplitudes unchanged ***BUT*** phases advance by $k\cdot\pi/2$.





Complex FS

Euler's notation:

$e^{-jt} = (e^{jt})^* = \cos(t) - j\cdot\sin(t)$ "phasor" $\cos(t) = \dfrac{e^{jt} + e^{-jt}}{2}$ $\sin(t) = \dfrac{e^{jt} - e^{-jt}}{2\cdot j}$

analysis

$$c_k = \frac{1}{T}\cdot\int_0^T s(t)\cdot e^{-jk\omega t}dt$$

synthesis

$$s(t) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{jk\omega t}$$

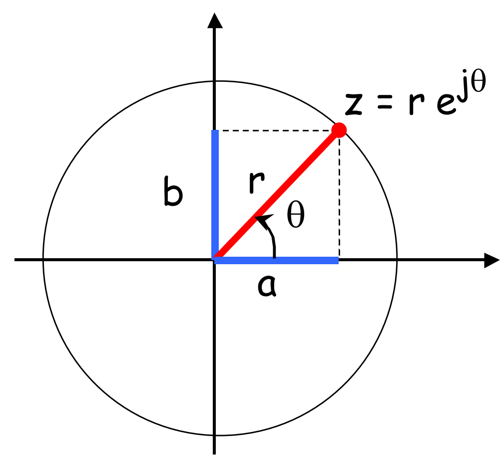
Complex form of FS (Laplace 1782). Harmonics c_k separated by $\Delta f = 1/T$ on frequency plot.

Note: $c_{-k} = (c_k)^*$

Link to FS real coeffs.

$$c_0 = a_0$$

$$c_k = \frac{1}{2}\cdot(a_k + j\cdot b_k) = \frac{1}{2}\cdot(a_{-k} - j\cdot b_{-k})$$



$r = \sqrt{a^2 + b^2}$

$\theta = \arctan(b/a)$

FS properties

	Time	Frequency
Homogeneity	$a \cdot s(t)$	$a \cdot S(k)$
Additivity	$s(t) + u(t)$	$S(k)+U(k)$
Linearity	$a \cdot s(t) + b \cdot u(t)$	$a \cdot S(k)+b \cdot U(k)$
Time reversal	$s(-t)$	$S(-k)$
Multiplication *	$s(t) \cdot u(t)$	$\sum_{m=-\infty}^{\infty} S(k-m)U(m)$
Convolution *	$\frac{1}{T} \cdot \int_0^T s(t-\bar{t}) \cdot u(\bar{t}) d\bar{t}$	$S(k) \cdot U(k)$
Time shifting	$s(t-\bar{t})$	$e^{-j\frac{2\pi k \cdot \bar{t}}{T}} \cdot S(k)$
Frequency shifting	$e^{+j\frac{2\pi m t}{T}} \cdot s(t)$	$S(k - m)$

*

FS - "oddities"

Orthonormal base

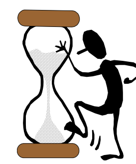
Fourier components $\{u_k\}$ form orthonormal base of signal space:

$u_k = (1/\sqrt{T}) \exp(jk\omega t)$ ($|k| = 0,1\ 2, \ldots +\infty$) **Def.:** Internal product $\otimes$: $u_k \otimes u_m = \int_0^T u_k \cdot u_m^* \, dt$

$u_k \otimes u_m = \delta_{k,m}$ (1 if k = m, 0 otherwise). (Remember $(e^{jt})^* = e^{-jt}$)

Then $c_k = (1/\sqrt{T})\ s(t) \otimes u_k$ i.e. $(1/\sqrt{T})$ times *projection* of signal s(t) on component u_k

Negative frequencies & time reversal



$k = -\infty, \ldots -2,-1,0,1,2, \ldots +\infty$, $\omega_k = k\omega$, $\phi_k = \omega_k t$, phasor turns anti-clockwise.

Negative k $\Rightarrow$ phasor turns clockwise (negative phase ϕ_k), equivalent to negative time t,

$\Rightarrow$ time reversal.

Careful: phases important when combining several signals!

FS - power

Average power W : $W = \frac{1}{T}\int_{o}^{T} |s(t)|^2 \, dt \equiv s(t) \otimes s(t)$

Parseval's Theorem

$$W = \sum_{k=-\infty}^{\infty} |c_k|^2 = a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}\left(a_k^2 + b_k^2\right)$$

- FS convergence ~1/k

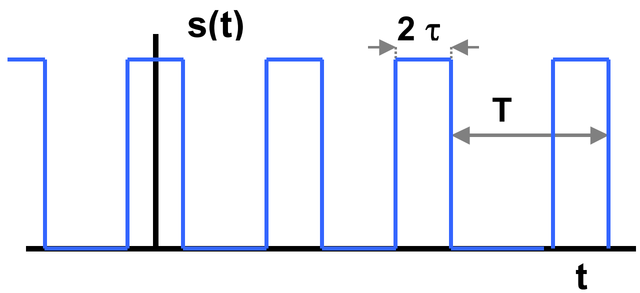
  $\Rightarrow$ *lower frequency terms*

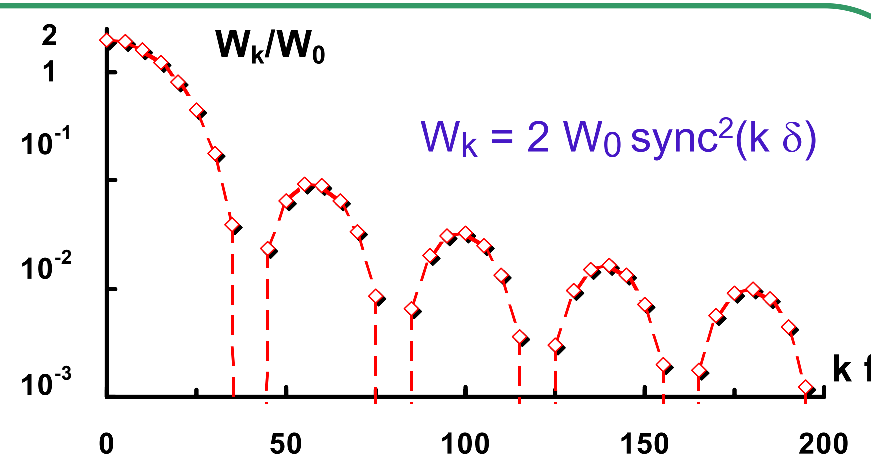
  $W_k = |c_k|^2$ carry most power.

- W_k vs. ω_k: Power density spectrum.

Example

Pulse train, duty cycle $\delta = 2\,\tau / T$





$W_k = 2\,W_0\,\text{sync}^2(k\,\delta)$

$b_k = 0 \quad a_0 = \delta\, s_{MAX}$

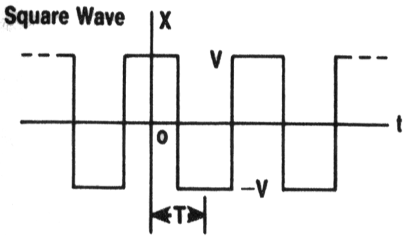
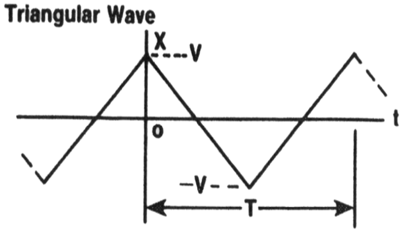
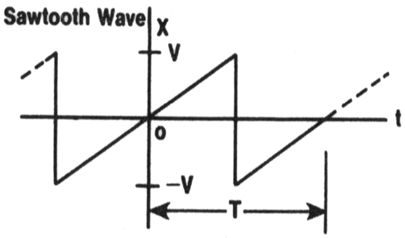
$a_k = 2\delta s_{MAX}\,\text{sync}(k\,\delta)$

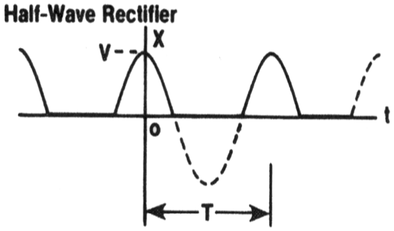
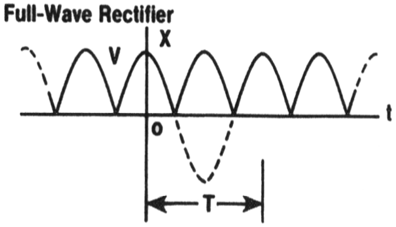
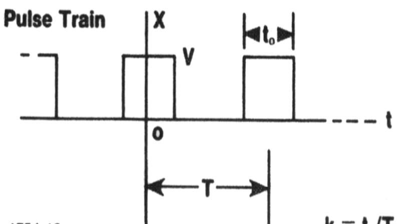
$W_0 = (\delta\, s_{MAX})^2$

$\text{sync}(u) = \sin(\pi\, u)/(\pi\, u)$

$$W = W_0 \cdot \left\{ 1 + \sum_{k=1}^{\infty} \frac{W_k}{W_0} \right\}$$

FS of main waveforms

Wave Shape	Fourier Series -- $\omega_o = 2\pi/T$
Square Wave 	$x(t) = \frac{4V}{\pi}(\cos\omega_o t - \frac{1}{3}\cos 3\omega_o t + \frac{1}{5}\cos 5\omega_o t - \frac{1}{7}\cos 7\omega_o t + \ldots)$
Triangular Wave 	$x(t) = \frac{8V}{\pi^2}(\cos\omega_o t + \frac{1}{9}\cos 3\omega_o t + \frac{1}{25}\cos 5\omega_o t + \ldots)$
Sawtooth Wave 	$x(t) = \frac{2V}{\pi}(\sin\omega_o t - \frac{1}{2}\sin 2\omega_o t + \frac{1}{3}\sin 3\omega_o t - \frac{1}{4}\sin 4\omega_o t + \ldots)$

Wave Shape	Fourier Series -- $\omega_o = 2\pi/T$
Half-Wave Rectifier 	$x(t) = \frac{V}{\pi}(1 + \frac{\pi}{2}\cos\omega_o t + \frac{2}{3}\cos 2\omega_o t - \frac{2}{15}\cos 4\omega_o t + \frac{2}{35}\cos 6\omega_o t - \ldots \ldots (-1)^{n/2+1}\frac{2}{n^2-1}\cos n\omega_o t \ldots)$ n even
Full-Wave Rectifier 	$x(t) = \frac{2V}{\pi}(1 + \frac{2}{3}\cos 2\omega_o t - \frac{2}{15}\cos 4\omega_o t + \frac{2}{35}\cos 6\omega_o t - \ldots \ldots (-1)^{n/2+1}\frac{2}{n^2-1}\cos n\omega_o t \ldots)$ n even
Pulse Train  1754-16 $k = t_o/T$	$x(t) = V[k + \frac{2}{\pi}(\sin k\pi \cos\omega_o t + \frac{1}{2}\sin 2k\pi \cos 2\omega_o t + \ldots \ldots + \frac{1}{n}\sin nk\pi \cos n\omega_o t + \ldots)]$ $k = t_o/T$

Slides adapted from ME Angoletta, CERN

Discrete Fourier Series (DFS)

Band-limited signal s[n], period = N.

DFS generate periodic c_k with same signal period

DFS defined as:

analysis

$$\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} s[n] \cdot e^{-j\frac{2\pi k n}{N}}$$

Note: $\tilde{c}_{k+N} = \tilde{c}_k \Leftrightarrow$ same period N

i.e. time periodicity propagates to frequencies!

synthesis

$$s[n] = \sum_{k=0}^{N-1} \tilde{c}_k \cdot e^{j\frac{2\pi k n}{N}}$$

Synthesis: finite sum $\Leftarrow$ band-limited s[n]

Orthogonality in DFS:

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi\, n(k-m)}{N}} = \delta_{k,m}$$

Kronecker's delta

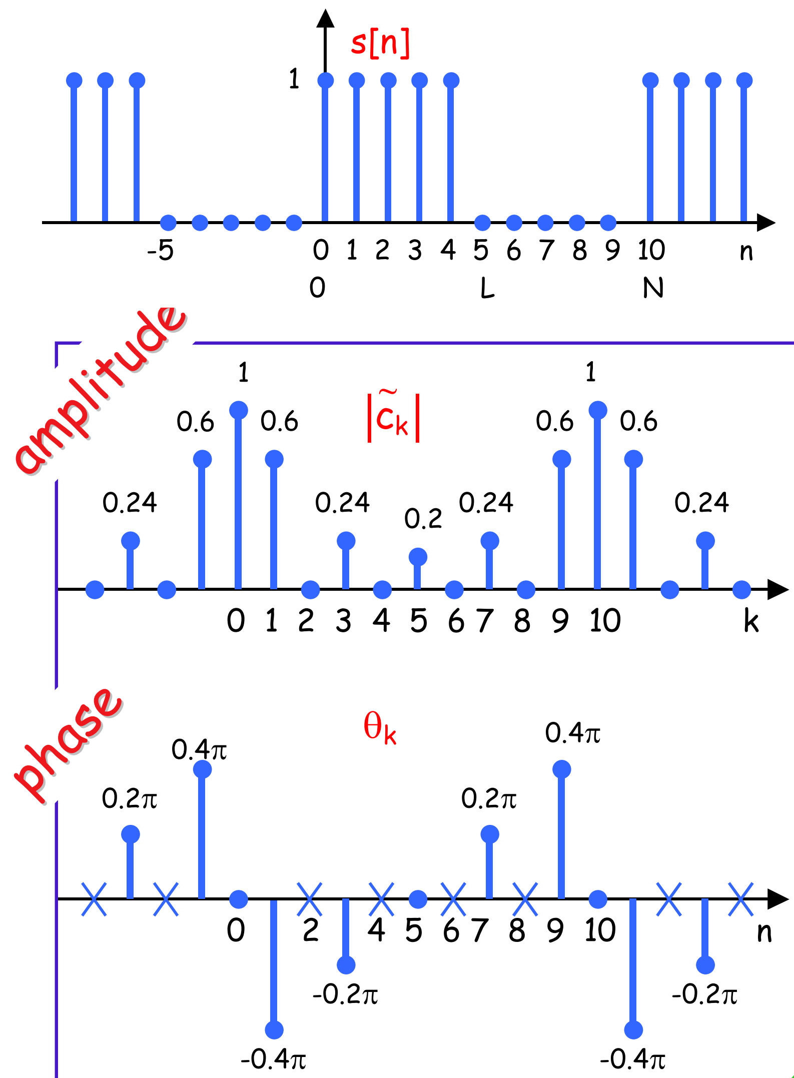
N consecutive samples of s[n] completely describe s in time or frequency domains.

DFS analysis

DFS of periodic discrete 1-Volt square-wave

s[n]: period N, duty factor L/N

$$\tilde{c}_k = \begin{cases} \dfrac{L}{N}, & k = 0, +N, \pm 2N, \ldots \\[2ex] \dfrac{e^{-j\frac{\pi k (L-1)}{N}}}{N} \cdot \dfrac{\sin\left(\frac{\pi kL}{N}\right)}{\sin\left(\frac{\pi k}{N}\right)}, & \text{otherwise} \end{cases}$$



Discrete signals ⇒ periodic frequency spectra.
Compare to continuous rectangular function (slide # 10, “FS analysis - 1”)

DFS properties

	Time	Frequency
Homogeneity	$a \cdot s[n]$	$a \cdot S(k)$
Additivity	$s[n] + u[n]$	$S(k)+U(k)$
Linearity	$a \cdot s[n] + b \cdot u[n]$	$a \cdot S(k)+b \cdot U(k)$
Multiplication *	$s[n] \cdot u[n]$	$\frac{1}{N} \cdot \sum_{h=0}^{N-1} S(h)U(k-h)$
Convolution *	$\sum_{m=0}^{N-1} s[m] \cdot u[n-m]$	$S(k) \cdot U(k)$
Time shifting	$s[n - m]$	$e^{-j\frac{2\pi k \cdot m}{T}} \cdot S(k)$
Frequency shifting	$e^{+j\frac{2\pi h t}{T}} \cdot s[n]$	$S(k - h)$

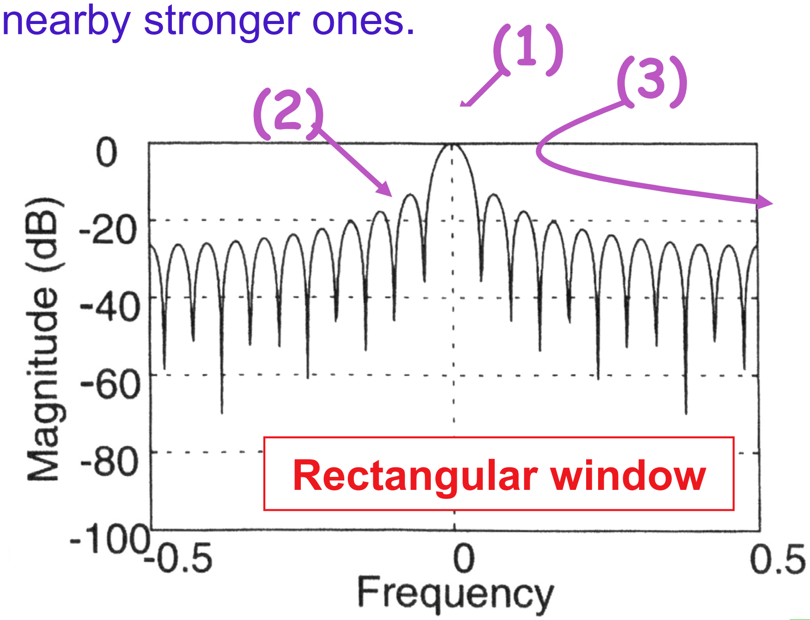
DFT – Window characteristics

- Finite discrete sequence ⇒ spectrum convoluted with rectangular window spectrum.
- Leakage amount depends on chosen window & on how signal fits into the window.

(1) **Resolution**: capability to distinguish different tones. Inversely proportional to main-lobe width. *Wish: as high as possible.*

(2) **Peak-sidelobe level**: maximum response outside the main lobe. Determines if small signals are hidden by nearby stronger ones. *Wish: as low as possible.*

(3) **Sidelobe roll-off**: sidelobe decay per decade. Trade-off with (2).

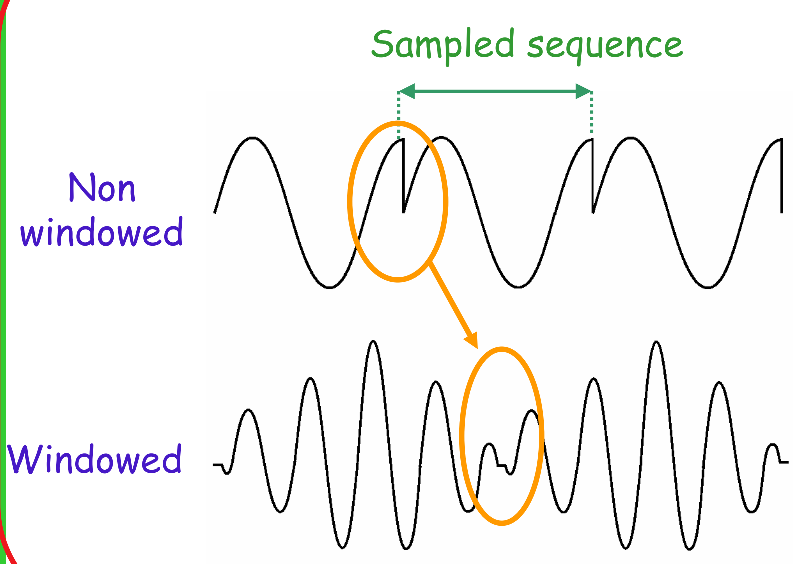


Several windows used (**application-dependent**): Hamming, Hanning, Blackman, Kaiser ...

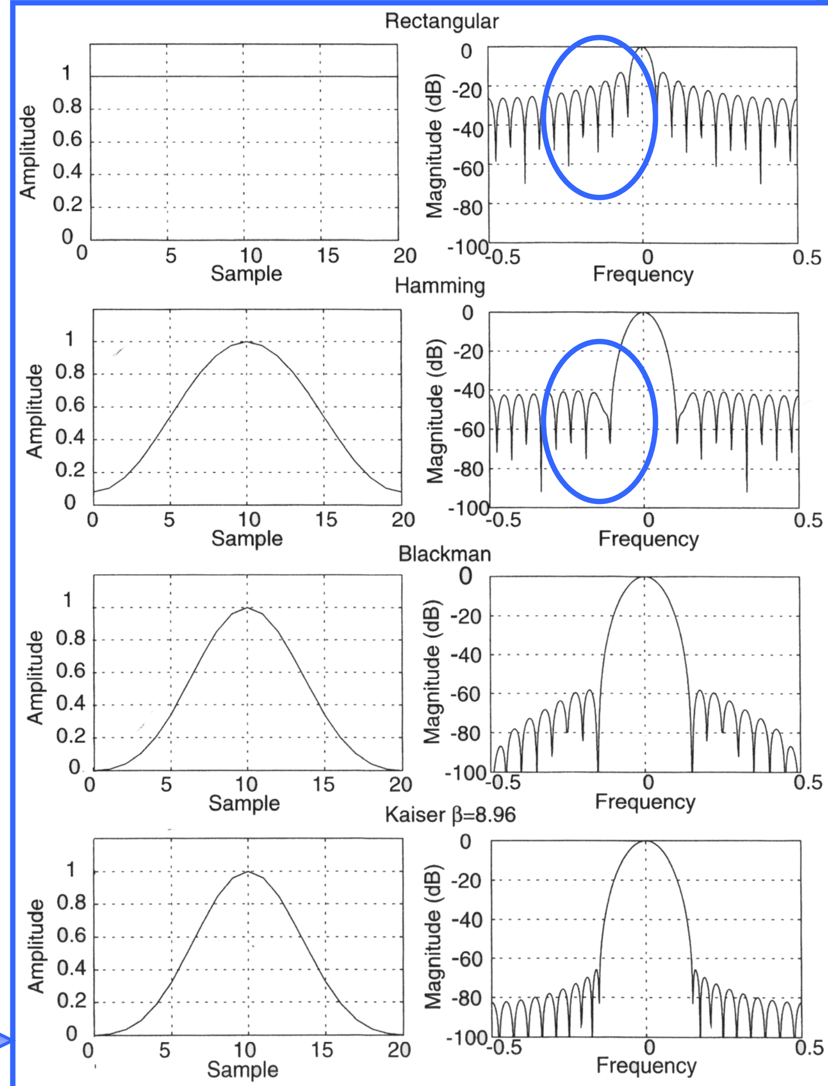
DFT of main windows

Windowing reduces leakage by minimising sidelobes magnitude.

In time it reduces end-points discontinuities.



Some window functions



DFT - Window choice

Common windows characteristics

Window type	-3 dB Main-lobe width [bins]	-6 dB Main-lobe width [bins]	Max sidelobe level [dB]	Sidelobe roll-off [dB/decade]
Rectangular	0.89	1.21	-13.2	20
Hamming	1.3	1.81	- 41.9	20
Hanning	1.44	2	- 31.6	60
Blackman	1.68	2.35	-58	60

Observed signal		Window wish list
Far & strong interfering components	⇒	high roll-off rate.
Near & strong interfering components	⇒	small max sidelobe level.
Accuracy measure of single tone	⇒	wide main-lobe

NB: Strong DC component can shadow nearby small signals. Remove it!

DFT - Window loss remedial

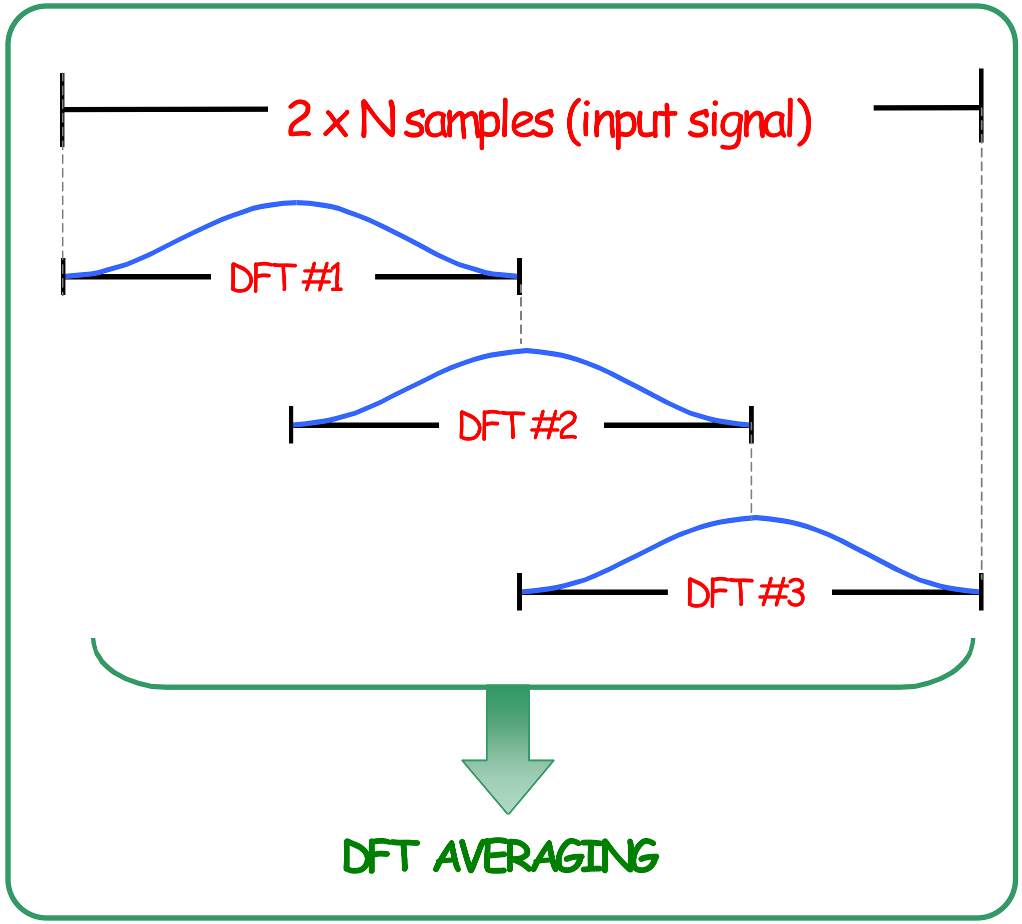
Smooth data-tapering windows cause information loss near edges.

Solution:

sliding (overlapping) DFTs.

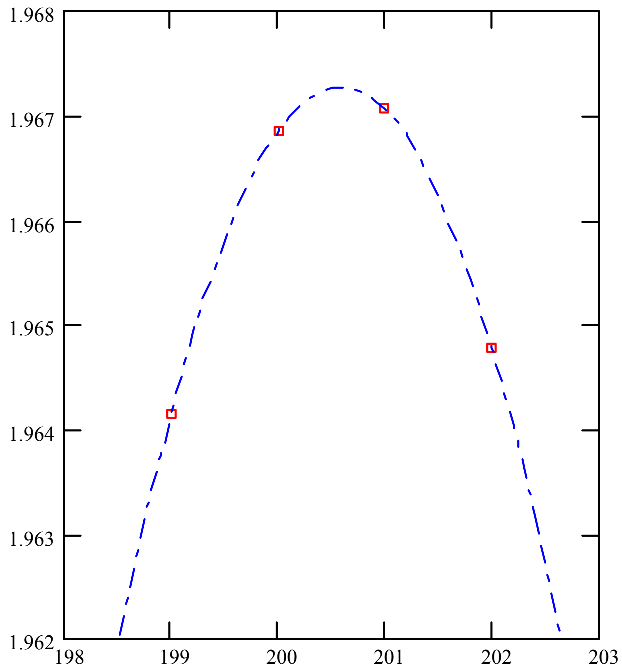
- Attenuated inputs get next window's full gain & leakage reduced.
- Usually 50% or 75% overlap (depends on main lobe width).

Drawback: increased total processing time.

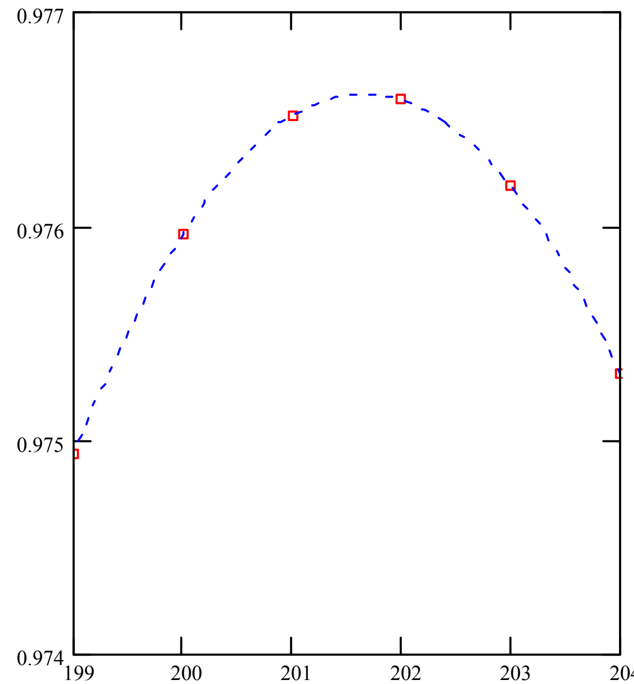


DFT - parabolic interpolation

Rectangular window



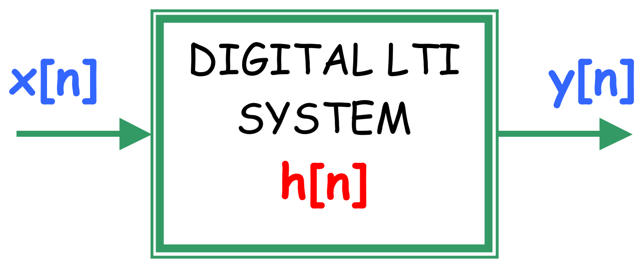
Hanning window

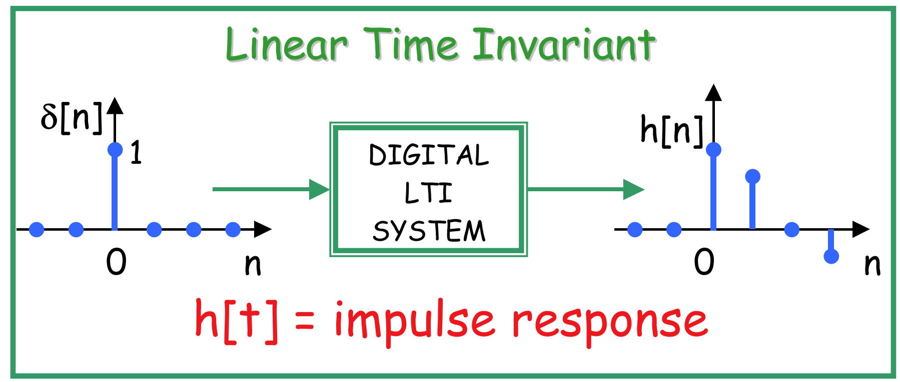


- Parabolic interpolation often enough to find position of peak (i.e. frequency).
- Other algorithms available depending on data.

Systems spectral analysis (hints)

System analysis: measure input-output relationship.





$$x[n] \leftrightarrow X(f) \qquad h[n] \leftrightarrow H(f)$$

$$y[n] = x[n] * h[n] = \sum_{m=0}^{\infty} x[n-m] \cdot h[m]$$

$$Y(f) = X(f) \cdot H(f)$$

y[n] predicted from { x[n], h[t] }

H(f) : LTI transfer function

Transfer function can be estimated by Y(f) / X(f)

Estimating H(f) (hints)

$G_{xx}(f) = X(f) \cdot X^*(f)$ Power Spectral Density of x[t] (FT of autocorrelation).

$G_{yx}(f) = Y(f) \cdot X^*(f)$ Cross Power Spectrum of x[t] & y[t] (FT of cross-correlation).

$$H(f) = \frac{Y(f)}{X(f)} = \frac{Y(f) \cdot X^*(f)}{X(f) \cdot X^*(f)} = \frac{G_{yx}}{G_{xx}}$$

Transfer Function (ex: beam !)

It is a check on H(f) validity!